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ON A CLASS OF UNIVERSAL MODULAR SEQUENCE SPACES

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ABSTRACT

The class of all subspaces of quotient spaces of $l_p \oplus l_r$ contains all separable Orlicz sequence spaces.

1. Introduction

Recently, Johnson and Zippin [2] proved that if a subspace of a quotient space of $l_p, p > 1$, has a basis, then it can be decomposed into the form $(\oplus G_n)_p$, where $\dim G_n < \infty$. It is natural to inquire whether this structure theorem can be generalized to the next simplest class of spaces, namely $l_p \oplus l_q$. The answer is no. Rosenthal [11] showed that if $1 < q < 2$, then the class of subspaces of quotients of $l_q \oplus l_2$ contains every Orlicz sequence space l_M, where

$$(*)\qquad \frac{M(x)}{x^q} \text{ is increasing, } \frac{M(x)}{x^2} \text{ is decreasing.}$$

In particular, the class of subspaces of quotients of $l_q \oplus l_2$ contains l_r for all $r \in [q, 2]$. So no structure theorem similar to that obtained by Johnson and Zippin is possible.

PROBLEM. Can Rosenthal's result be generalized to the case $l_q \oplus l_s$, where $1 \leqq q < s < \infty$?

The purpose of this paper is to answer the above problem in the affirmative. In order to analyze this problem, we have to describe Rosenthal's result more fully. In [10], he obtained an $\mathscr{L}_p$-space $X_p, p > 2$, which is spanned by a sequence of independent random variables in L_p. X_p is also isomorphic to a subspace of $l_p \oplus l_2$. Let $X_q = X_p^*$, where $1/p + 1/q = 1$. Then every Orlicz sequence space l_M, where M satisfies $(*)$, can be imbedded into X_q [11], and thus is a subspace of a quotient of $l_q \oplus l_2$. The proofs of these results use

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probabilistic methods, and since Khintchine's Inequality is used in a nontrivial way, we cannot replace 2 by s.

However, there is another way to look at the spaces $X_p, p > 2$. X_p has an unconditional basis $\{e_n\}$, and $\Sigma x_n e_n$ converges if and only if $\Sigma \max\{|x_n|^p, w_n^2 x_n^2\} < \infty$, where $w = \{w_n\}$ is a sequence of positive numbers satisfying $\lim w_n = 0$ and $\Sigma w_n^{2p/(p-2)} = \infty$. Put $M_n(x) = \max\{x^p, w_n^2 x^2\}$ for $x \geqq 0$. Then M_n is an Orlicz function, and so X_p is a modular sequence space (see Sec. 2 for definitions). We can now generalize X_p to a space $X_{p,r}$, where $\infty > p > r \geqq 1$. We simply define $X_{p,r}$ to be the modular sequence space $l\{M_n\}$, where $M_n(x) = \max\{x^p, w_n^r x^r\}$. Our aim is to prove that if $Y_{q,s} = X_{p,r}^*$, where $1/p + 1/q = 1$, $r > 1$, and $1/r + 1/s = 1$, then $Y_{q,s}$ contains every Orlicz sequence space l_M, where

$$(**) \qquad \frac{M(x)}{x^q} \text{ is increasing, } \frac{M(x)}{x^s} \text{ is decreasing.}$$

As $X_{p,r}$ is a subspace of $l_p \oplus l_r$, this would solve the problem.

As is apparent from the above, we shall use the techniques of modular sequence spaces to solve the problem. Detailed accounts of the properties of such spaces can be found in [13] and [14]. A summary of the important results used in this paper is given in Sec. 2, mostly without proofs. Section 3 is devoted to the construction of the spaces $X_{p,r}$, while the main results are proved in Sec. 4. We prove that $Y_{q,s}$ is universal in the class of modular sequence spaces $l\{M_n\}$, where M_n satisfies (**) for all but finitely many n's. In particular, $Y_{q,s}$ contains all l_M, where M satisfies (**). Thus the problem is solved.

As byproducts, we also obtain the following results. The first is a representation of Orlicz functions. Suppose M is an Orlicz function satisfying (**). Then M is equivalent to the uniform limit of functions of the form $\Sigma_{i=k_n+1}^{k_{n+1}} N_i(a_i x)$, where $k_1 < k_2 < \cdots$ are positive integers, $a_i \geqq 0$, and $N_i(x) = \min\{w_i^{-s} x^s, x^q\}$, with $w_i > 0$ and $\lim w_i = 0$. The other result is an answer in the affirmative to a question raised by Lindenstrauss: Does there exist a nonsymmetric unconditional basis so that the span of every block of constant coefficients is complemented? The answer is provided by the natural basis of $X_{\infty,r}$ which has this property.

2. Definitions, notations and preliminaries

Let $M : [0,\infty) \to [0,\infty)$ be a continuous *strictly increasing* function satisfying $M(0) = 0$ and $\lim_{x\to\infty} M(x) = \infty$. M is called a *φ-function*. Note that in [6] and [8]

M is only required to be monotonic increasing, and $M(x) > 0$ for all $x > 0$. However, nothing is gained by this more general condition as far as this paper is concerned. In fact by restricting to strictly increasing functions, we now have $M^{-1}: [0,\infty) \to [0,\infty)$, and M^{-1} is also a φ-function. If M is convex, then M is called an *Orlicz function.*

DEFINITION. Let $0 \leqq r < p \leqq \infty$. A φ-function M is said to be *between p and r* if $(M(x))/x^r$ is increasing on $(0,\infty)$, and $(M(x))/x^p$ is decreasing on $(0,\infty)$ for $p < \infty$. Following the terminology of [1], we let $K(p,r)$ be the set of all φ-functions between p and r.

Note that every φ-function is in $K(\infty,0)$, every Orlicz function is in $K(\infty,1)$, and every concave φ-function is in $K(1,0)$. If $p_1 \geqq p_2$, and $r_1 \leqq r_2$, then $K(p_2,r_2) \subset K(p_1,r_1)$.

The following theorem of Matuszewska is of fundamental importance. As it is stated in a different form in 2.7 of [7], we give the proof here.

THEOREM 2.1. (Matuszewska) (a) *Let $M \in K(p,1)$, where $p \leqq \infty$. Then there exists an Orlicz function $N \in K(p,1)$, with continuous second derivative, such that*

$$M\left(\frac{x}{4}\right) \leqq N(x) \leqq M(x) \text{ for all } x \geqq 0.$$

(b) *Let $M \in K(1,r)$, where $r \geqq 0$. Then there exists a concave φ-function $N \in K(1,r)$, with continuous second derivative, such that*

$$M(x) \leqq N(x) \leqq 4M(x) \text{ for all } x \geqq 0.$$

PROOF. (a) Let $F(x) = \int_0^x (M(t))/t\,dt$. As $M \in K(p,1)$, $(M(t))/t$ is increasing on $(0,\infty)$. Hence F is an Orlicz function, and F has a continuous derivative. It is not hard to show that $F \in K(p,1)$. By monotonicity of $(M(t))/t$, we have

$$M(x) = x\frac{M(x)}{x} \geqq \int_0^x \frac{M(t)}{t}dt = F(x)$$

$$\geqq \int_{\frac{x}{2}}^x \frac{M(t)}{t}dt \geqq \frac{x}{2}\frac{M\left(\frac{x}{2}\right)}{\frac{x}{2}} = M\left(\frac{x}{2}\right).$$

So we have $F \in K(p,1)$, with continuous derivative, such that

$$M(x) \geqq F(x) \geqq M\left(\frac{x}{2}\right) \text{ for all } x \geqq 0.$$

Now apply the above to F instead of M, and we obtain an Orlicz function $N \in K(p,1)$, with continuous second derivative, such that

$$F(x) \geqq N(x) \geqq F\left(\frac{x}{2}\right) \text{ for all } x \geqq 0,$$

i.e.

$$M(x) \geqq N(x) \geqq M\left(\frac{x}{4}\right) \text{ for all } x \geqq 0.$$

(b) Consider the φ-function M^{-1}. It is easy to show that $M^{-1} \in K(r^{-1},1)$. Hence by (a), we have an Orlicz function $F \in K(r^{-1},1)$, with continuous second derivative, such that

$$M^{-1}(x) \geqq F(x) \geqq M^{-1}\left(\frac{x}{4}\right) \text{ for all } x \geqq 0.$$

Let $N = F^{-1}$. Then N is a concave φ-function between 1 and r, with continuous second derivative, and satisfies

$$M(x) \leqq N(x) \leqq 4M(x) \text{ for all } x \geqq 0. \qquad \text{Q.E.D.}$$

Definition. Let M, N be φ-functions. M is said to be *equivalent to* N $(M \sim N)$ if there exist positive numbers a, b, K, L, x_0 such that

$$KM(ax) \leqq N(x) \leqq LM(bx)$$

for all $x \in [0, x_0]$.

Corollary 2.2. *Let $M \in K(p,r)$, where $r > 0$. Suppose M is not equivalent to x^r. Then there exists an Orlicz function $F \in K(pr^{-1},1)$, with continuous second derivative, such that*

(a) *$M(x^{1/r}/4) \leqq F(x) \leqq M(x^{1/r})$ for all $x \geqq 0$,*

(b) *$F'(x)$ is strictly increasing and continuous,*

(c) *$(F(x))/x$ is a φ-function.*

Thus we have a function $N \in K(p,r)$, with continuous second derivative, such that

$$M\left(\frac{x}{4}\right) \leqq N(x) \leqq M(x) \quad \textit{for all} \quad x \geqq 0,$$

and $(N(x))/x^r$ is a φ-function.

Proof. Let $G(x) = M(x^{1/r})$. Then $G \in K(pr^{-1},1)$. By Matuszewska's Theorem, we have an Orlicz function $F \in K(pr^{-1},1)$, with continuous second derivative, such that

$$G\left(\frac{x}{4}\right) \leqq F(x) \leqq G(x) \text{ for all } x > 0.$$

M is not equivalent to x^r iff G is not equivalent to x. As F is equivalent to G, F is not equivalent to x. F is convex, so F' and $(F(x))/x$ must be strictly increasing, and $\lim_{x\to 0}(F(x))/x = 0$. Thus $(F(x))/x$ is a φ-function. Putting $N(x) = F(x^r)$, we have the desired result. Q.E.D.

We now define φ-spaces and modular sequence spaces. Let $\{M_n\}$ be a sequence of φ-functions. Let $l\{M_n\}$ be the set of all sequences $\{x_n\}$ satisfying $\sum M_n(|x_n|/t) < \infty$ for some $t > 0$. Let

$$\|\{x_n\}\| = \inf\left\{t > 0 : \sum M_n\left(\frac{|x_n|}{t}\right) \leqq t\right\}.$$

Then $\|\cdot\|$ is a quasinorm on $l\{M_n\}$, and $(l\{M_n\}, \|\cdot\|)$ is an F-space. We call such kind of spaces *φ-spaces*. If all the M_n's are equal, we have the generalized Orlicz sequence space considered in [6] and [8].

If all the M_n's are Orlicz functions, then $l\{M_n\}$ can be normed by

$$|||\{x_n\}||| = \inf\left\{t > 0 : \sum M_n\left(\frac{|x_n|}{t}\right) \leqq 1\right\}.$$

$(l\{M_n\}, |||\cdot|||)$ is called a *modular sequence space*. The spaces $(l\{M_n\}, \|\cdot\|)$ and $(l\{M_n\}, |||\cdot|||)$ are isomorphic under the identity map. So whenever we are considering sequences of Orlicz functions, $\|\cdot\|$ will actually denote $|||\cdot|||$, unless otherwise stated.

A sequence of φ-functions $\{M_n\}$ is said to be *normalized* if $M_n(1) = 1$ for all n. If $\{M_n\}$ is not normalized, let $N_n(x) = M_n(a_n x)$, where $M_n(a_n) = 1$. Then $\{N_n\}$ is normalized and the map $T: l\{N_n\} \to l\{M_n\}$ given by $T\{x_n\} = \{a_n x_n\}$ is an isomorphism. So if we are only considering the linear topological properties of $l\{M_n\}$, we can always assume $\{M_n\}$ to be normalized.

The most important concept connected with φ-spaces is that of equivalence.

DEFINITION. Let $\{M_n\}$, $\{N_n\}$ be sequences of φ-functions, $\{M_n\}$ is said to be *equivalent* to $\{N_n\}$ ($\{M_n\} \sim \{N_n\}$) if $l\{M_n\} = l\{N_n\}$ as sets.

It is easy to show, using the closed graph theorem, that if $\{M_n\} \sim \{N_m\}$, then the identity map $I: l\{M_n\} \to l\{N_n\}$ is an isomorphism.

PROPOSITION 2.3. *Let $\{M_n\}$, $\{N_n\}$ be sequences of φ-functions satisfying one of the following conditions:*

(a) *There exist x_1, x_2 such that* $\inf M_n(x_1) > 0$, $\inf N_n(x_2) > 0$, *and there exist $K, L, a, b, x_o > 0$, $n_0 \in Z^+$ such that for all $n > n_0$ and $x \in [0, x_0]$,*

$$KM_n(ax) \leqq N_n(x) \leqq LM_n(bx).$$

(b) *There exist* x_1, x_2 *such that* $\inf M_n(x_1) > 0$ *and* $\inf N_n(x_2) > 0$, *and for some* $\alpha \geqq \max\{x_1, x_2\}$,

$$\sum_{n=1}^{\infty} \sup_{x \in [0,\alpha]} |M_n(x) - N_n(x)| < \infty.$$

Then $\{M_n\} \sim \{N_n\}$.

Definition. Let $\{M_n\}$ be a sequence of φ-functions, and let $0 \leqq r < p \leqq \infty$. We say that $\{M_n\}$ is between p and r, if $M_n \in K(p, r)$ for all but finitely many n's. We again denote the set of all $\{M_n\}$ between p and r by $K(p, r)$.

Proposition 2.4. *Let* $\{M_n\}$ *be a sequence of* φ*-functions. Then the following are equivalent*:

(a) $\{M_n\} \in K(p, r)$.

(b) $\{M_n(x^t)\} \in K(pt, rt)$ *for* $\binom{all}{some}$ $t \in (0, \infty)$.

(c) *There exists* $n_0 > 0$ *such that for all* $\lambda \in [0, 1]$, $x \geqq 0$ *and* $n > n_0$,

$$\lambda^p M_n(x) \leqq M_n(\lambda x) \leqq \lambda^r M_n(x).$$

If M_n *is absolutely continuous for all* n, *then the above are equivalent to*:

(d) *There exists* n_0 *such that*

$$r \leqq \frac{xM_n'(x)}{M_n(x)} \leqq p$$

for all $n > n_0$ *and* $x > 0$ *such that* $M_n'(x)$ *exists.*

Theorem 2.5. (Lindberg) *Let* $\{M_n\}$ *be a normalized sequence of* φ*-functions (Orlicz functions), and* $\{M_n\} \in K(p, r)$ *for some* $0 < r < p < \infty$. *Then*

(a) $\{M_n\}$ *is uniformly equicontinuous and uniformly bounded on* $[0, 1]$.

(b) *There exists a subsequence* $\{M_{n_k}\}$ *and a* φ*-function (Orlicz function)* $M \in K(p, r)$ *such that*

$$\sum_{k=1}^{\infty} \sup_{x \in [0,1]} |M_{n_k}(x) - M(x)| < \infty.$$

(c) $l\{M_n\}$ *contains a complemented subspace isomorphic to* l_M.

If $\{M_n\} \in K(p, r)$ for some $p < \infty$, it is not hard to show that the unit vectors of $l\{M_n\}$ form an unconditional basis. If $M \in K(p, r)$ for some $p < \infty$, then the unit vectors of l_M form a symmetric basis. Now suppose l_M and $l\{M_n\}$ are both Banach spaces, and l_M is isomorphic to a subspace of $l\{M_n\}$. Then by the Bessaga-Pelczynski Theorem, the unit vectors of l_M is equivalent to a block basis of $l\{M_n\}$. We can assume $\{M_n\}$, and the bases, to be normalized. Let $\{e_n\}$, $\{f_n\}$ be the unit vector bases of $l_M, l\{M_n\}$, respectively. Then we have $a_n \geqq 0$, and natural numbers $j_1 < j_2 < \cdots$ such that $\{e_n\}$ is equivalent to the normalized block basis $\{\sum_{j=j_k+1}^{j_{k+1}} a_n f_n\}_{k=1}^{\infty}$. It is not hard to show that this block basis is equivalent to the unit vector basis of the modular sequence space $l\{N_k\}$, where

$$N_k(x) = \sum_{n=j_k+1}^{j_{k+1}} M_n(a_n x). \tag{1}$$

By passing to a subsequence of $\{N_k\}$, and using Theorem 2.5, we have

$$\sum_{k=1}^{\infty} \sup_{x\in[0,1]} |M(x) - N_k(x)| < \infty.$$

We thus have the following proposition:

PROPOSITION 2.6. (Lindberg) (a) *Let $\{M_n\}$ be a normalized sequence of Orlicz functions between p and r, where $1 \leqq r < p < \infty$. Suppose M is an Orlicz function such that l_M is isomorphic to a subspace of $l\{M_n\}$. Then there exists a sequence $\{N_k\}$, satisfying* (1), *such that*

$$\sum_{k=1}^{\infty} \sup_{x\in[0,1]} |M(x) - N_k(x)| < \infty.$$

(b) *Let $\{M_n\}$, M be as in* (a). *Then M is equivalent to an Orlicz function in* $K(p,r)$.

Let $\{M_n\}$ be a sequence of Orlicz functions, none of which is equivalent to the function x. Then we can form the Young complement M_n^* of M_n. i.e. M_n^* is an Orlicz function satisfying

$$M_n^{*\prime}(y) = \sup_{x \leqq y} M_n'(x),$$

where M_n' is the right derivative.

Let $c\{M_n\}$ be the set

$$\left\{\{x_n\} \in l\{M_n\} : \sum M_n\left(\frac{|x_n|}{t}\right) < \infty \text{ for all } t > 0\right\}.$$

Then $c\{M_n\}$ is a closed subspace of $l\{M_n\}$, and the unit vectors form an unconditional basis of $c\{M_n\}$.

PROPOSITION 2.7. *Let* $\{M_n\}$ *be a sequence of Orlicz functions.*

(a) $c\{M_n\} = l\{M_n\}$ *if* $\{M_n\} \in K(p,1)$ *for some* $p < \infty$.

(b) *Suppose none of the* M_n*'s is equivalent to* x. *Then* $c\{M_n\}^*$ *is isomorphic to* $l\{M_n^*\}$.

(c) *Let* $\{M_n\} \in K(p,1)$ *for some* $p < \infty$, *and none of the* M_n*'s is equivalent to* x. *Then* $l\{M_n\}^*$ *is isomorphic to* $l\{M_n^*\}$.

3. The spaces $X_{p,q}$

We are going to generalize the spaces $X_p, p > 2$, of [10] in this section. We then show that our generalizations are modular sequence spaces. A by-product of our considerations of the properties of these spaces is that we have answered in the affirmative a question raised by Lindenstrauss: Does there exist a non-symmetric basis such that every block of constant coefficients spans a complemented subspace?

Let $\infty > p > r \geqq 1$, and let $\{f_n\}$, $\{g_n\}$ be the unit vector bases of l_p and l_r respectively. Let $w = \{w_n\}$ be a sequence of positive numbers, and let $e_n = f_n + w_n g_n$. We define $X_{p,r,w}$ to be span $\{e_n\}$ in $(l_p \oplus l_r)_\infty$. $\{e_n\}$ is called the *natural basis* of $X_{p,r,w}$. If we replace l_p by c_0 in the above definition, we obtain a subspace of $(c_0 \oplus l_r)_\infty$, denoted by $X_{\infty,r,w}$.

It is not hard to see that $\{e_n\}$ is equivalent to the unit vector basis of l_p if $\Sigma w_n^{pr/(p-r)} < \infty$ (and equivalent to the unit vector basis of c_0 if $\Sigma w_n^r < \infty$ in the case $p = \infty$). On the other hand, if $\inf w_n > 0$, then $X_{p,r,w}$ is isomorphic to l_r. We thus have the following cases:

Case (i). $\inf w_n > 0$ and $X_{p,r,w} \sim l_r$.

Case (ii). $\Sigma w_n^{pr/(p-r)} < \infty$ and $X_{p,r,w} \sim l_p$ (or c_0).

Case (iii). $\{n \in \mathbf{Z}^+ : w_n \geqq \epsilon\}$ is infinite, and

$$\sum_{w_n < \epsilon} w_n^{pr/(p-r)} < \infty.$$

Then $X_{p,r,w} \sim l_p \oplus l_r$ (or $c_0 \oplus l_r$ for $p = \infty$).

Case (iv). $\inf w_n = 0$ and $\Sigma_{w_n < \epsilon} w_n^{pr/(p-r)} = \infty$ for all $\epsilon > 0$.

The simplest case for which (iv) can occur is when

$$\lim w_n = 0 \text{ and } \sum w_n^{pr/(p-r)} = \infty. \qquad (2)$$
$$\left(\sum w_n^r = \infty \text{ for } p = \infty.\right)$$

Our aim is to show that if w, w' satisfy (2), then $X_{p,r,w} \sim X_{p,r,w'}$.

PROPOSITION 3.1. *Let $\infty \geqq p > r \geqq 1$, and let $\{e_n\}$ be the natural basis of $X_{p,r,w}$. Suppose $\{E_j\}$ is a family of disjoint finite subsets of Z^+. Let*

$$(3)\qquad \begin{cases} h_j = \sum_{n\in E_j} w_n^{r/(p-r)} e_n & \left(= \sum_{n\in E_j} e_n \text{ for } p=\infty\right) \\ \tilde{h}_j = h_j\left(\sum_{m\in E_j} w_m^{pr/(p-r)}\right)^{-1/p} & (= h_j \text{ for } p=\infty) \\ \beta_j = \left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{(p-r)/pr} & \left(= \left(\sum_{n\in E_j} w_n^r\right)^{1/r} \text{ for } p=\infty\right). \end{cases}$$

Then the block basis $\{\tilde{h}_j\}$ is isometrically equivalent to the natural basis of $X_{p,r,\beta}$, where $\beta = \{\beta_j\}$, and there exists a projection of norm 1 of $X_{p,r,w}$ onto the closed linear span of $\{\tilde{h}_j\}$ in $X_{p,r,w}$.

PROOF. The proof is a generalization of that of Lemma 7 in [10]. First consider the case $p<\infty$. Then

$$\left\|\sum \lambda_j \tilde{h}_j\right\| = \max\left\{\left[\sum_{j=1}^{\infty}\sum_{n\in E_j}|\lambda_j|^p w_n^{pr/(p-r)}\left(\sum_{m\in E_j} w_m^{pr/(p-r)}\right)^{-1}\right]^{1/p},\right.$$

$$\left.\left[\sum_{j=1}^{\infty}\sum_{n\in E_j}|\lambda_j|^r w_n^{pr/(p-r)}\left(\sum_{m\in E_j} w_m^{pr/(p-r)}\right)^{-r/p}\right]^{1/r}\right\}$$

$$= \max\left\{\left(\sum_{j=1}^{\infty}|\lambda_j|^p\right)^{1/p},\ \left(\sum_{j=1}^{\infty}|\lambda_j|^r\beta_j^r\right)^{1/r}\right\}.$$

So $\{\tilde{h}_j\}$ is isometrically equivalent to the natural basis of $X_{p,r,\beta}$.

We now define $P: X_{p,r,w} \to [\tilde{h}_j]$, with $\|P\|=1$. Let $k \in Z^+$. Define

$$P\left(\sum_{n=1}^{k}\lambda_n e_n\right) = \sum_{j=1}^{\infty}\left(\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right)\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{-1} h_j.$$

The sum on the right is a finite sum. So P is well defined.

$$\left\|P\left(\sum_{n=1}^{k}\lambda_n e_n\right)\right\| = \max\left\{\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right|^p\left|\sum_{n\in E_j} w_n^{pr/(p-r)}\right|^{-p}\sum_{m\in E_j} w_m^{pr/(p-r)}\right]^{1/p},\right.$$

$$\left.\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right|^r\left|\sum_{n\in E_j} w_n^{pr/(p-r)}\right|^{-r}\sum_{m\in E_j} w_m^{pr/(p-r)}\right]^{1/r}\right\}$$

$$= \max\left\{\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right|^p\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{1-p}\right]^{1/p},\right.$$

$$\left.\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right|^r\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{1-r}\right]^{1/r}\right\}$$

$$\leqq \max\left\{\left[\sum_{j=1}^{\infty}\left(\sum_{n\in E_j}|\lambda_n|^p\right)\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{p-1}\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{1-p}\right]^{1/p},\right.$$

$$\left.\left[\sum_{j=1}^{\infty}\left(\sum_{n\in E_j} w_n^r|\lambda_n|^r\right)\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{r-1}\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{1-r}\right]^{1/r}\right\}$$

$$=\left\|\sum_{j=1}^{\infty}\sum_{n\in E_j}\lambda_n e_n\right\|=\left\|\sum_{n=1}^{k}\lambda_n e_n\right\|.$$

It is clear that we can extend P by

$$P\left(\sum_{n=1}^{\infty}\lambda_n e_n\right)=\sum_{j=1}^{\infty}\left(\sum_{n\in E_j}\lambda_n w_n^{(pr-r)/(p-r)}\right)\left(\sum_{n\in E_j} w_n^{pr/(p-r)}\right)^{-1} h_j$$

and we have

$$\left\|P\left(\sum_{n=1}^{\infty}\lambda_n e_n\right)\right\|\leqq\left\|\sum_{n=1}^{\infty}\lambda_n e_n\right\|.$$

It is obvious that P is a projection, and $\|P\|=1$.

Now consider the case $p=\infty$. Then

$$\left\|\sum\lambda_j\tilde{h}_j\right\|=\max\left\{\sup|\lambda_j|,\left[\sum_{j=1}^{\infty}\left(\sum_{n\in E_j}|\lambda_j|^r w_n^r\right)\right]^{1/r}\right\}$$

$$=\max\left\{\sup|\lambda_j|,\left(\sum_{j=1}^{\infty}|\lambda_j|^r\beta_j^r\right)^{1/r}\right\}.$$

So $\{\tilde{h}_j\}$ is isometrically equivalent to the natural basis of $X_{\infty,r,\beta}$.

As in the case $p<\infty$, we define

$$P\left(\sum\lambda_n e_n\right)=\sum_{j=1}^{\infty}\left(\sum_{n\in E_j}\lambda_n w_n^r\right)\left(\sum_{n\in E_j} w_n^r\right)^{-1} h_j.$$

Then P is a projection, and

$$\left\|P\left(\sum\lambda_n e_n\right)\right\|=\max\left\{\sup_j\left|\sum_{n\in E_j}\lambda_n w_n^r\right|\left(\sum_{n\in E_j} w_n^r\right)^{-1},\right.$$

$$\left.\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^r\right|^r\left(\sum_{n\in E_j} w_n^r\right)^{-r}\left(\sum_{m\in E_j} w_m^r\right)\right]^{1/r}\right\}$$

$$\leqq\max\left\{\sup_j\sup_{n\in E_j}|\lambda_n|,\left[\sum_{j=1}^{\infty}\left|\sum_{n\in E_j}\lambda_n w_n^r\right|^r\left(\sum_{n\in E_j} w_n^r\right)^{1-r}\right]^{1/r}\right\}$$

$$\leqq\max\left\{\sup_n|\lambda_n|,\left[\sum_{j=1}^{\infty}\left(\sum_{n\in E_j}|\lambda_n|^r w_n^r\right)\left(\sum_{n\in E_j} w_n^r\right)^{r-1}\left(\sum_{n\in E_j} w_n^r\right)^{1-r}\right]^{1/r}\right\}$$

$$\leqq\max\left\{\sup_n|\lambda_n|,\left(\sum_{n=1}^{\infty}|\lambda_n|^r w_n^r\right)^{1/r}\right\}=\left\|\sum\lambda_n e_n\right\|.$$

Q.E.D.

COROLLARY 3.2. *Suppose $w = \{w_n\}$ satisfies (2). Then $X_{p,r,w}$ contains complemented subspaces isomorphic to l_p and l_r (c_0 for $p = \infty$).*

PROOF. For l_p or c_o, just choose a subsequence $\{w_{n_k}\}$ such that $\Sigma_k w_{n_k}^{pr/(p-r)} < \infty$. Then $\{e_{n_k}\}$ is equivalent to the natural basis of $X_{p,r,\{w_{nk}\}}$, and so is equivalent to the unit vector basis of l_p or c_0.

For l_r, choose E_j such that $\beta_j = \Sigma_{n \in E_j} w_n^{pr/(p-r)} \geqq 1$. This is possible, since $\Sigma_{n=1}^{\infty} w_n^{pr/(p-r)} = \infty$. Let $\{\tilde{h}_j\}$ be as in Proposition 3.1. Then the subspace spanned by $\{\tilde{h}_j\}$ is isometric to $X_{p,r,\beta}$, which is isomorphic to l_r.

REMARK. Proposition 3.1 answers a question raised by Lindenstrauss: Is there a nonsymmetric unconditional basis such that every block basis of constant coefficients spans a completed subspace? The answer is yes, since the natural basis of $X_{\infty,r,w}$ is such an example. The basis is not symmetric, since it has a subsequence equivalent to the unit vector basis of c_0, which is clearly not equivalent to the natural basis of $X_{\infty,r,w}$.

THEOREM 3.3. *Suppose w, w' satisfy (2). Then $X_{p,r,w}$ and $X_{p,r,w'}$ are isomorphic. Moreover, suppose $\{e_n\}$, $\{e'_j\}$ are the respective natural bases. Then $\{e'_j\}$ is equivalent to a block basis $\{\tilde{h}_j\}$ of $\{e_n\}$, where $\{h_j\}$ is as in (3). Thus the natural basis of $X_{p,r,w'}$ is equivalent to a block basis of $\{e_n\}$, whose span is complemented in $X_{p,r,w}$.*

PROOF. The same as that of Theorem 13 in [10].

From now on, we denote $X_{p,r,w}$ by $X_{p,r}$ if w satisfies (2). We are going to show that the natural basis of $X_{p,r}$ is a modular basis.

First consider the case $\infty > p > r \geqq 1$. Let $w = \{w_n\}$ satisfy (2). Since $w_n \to 0$, we may assume $w_n \leqq 1$ for all n. Define

$$M_{w,n}(x) = \max\{x^p, w_n^r x^r\}$$
$$= \begin{cases} w_n^r x^r & x \in [0, w_n^{r/(p-r)}] \\ x^p & x \geqq w_n^{r/(p-r)}. \end{cases}$$

It is clear that $M_{w,n}$ is an Orlicz function in $K(p,r)$. If there is no danger of confusion, we shall write M_n for $M_{w,n}$. Note that $\Sigma M_n(|x_n|) < \infty$ if and only if $\max\{(\Sigma |x_n|^p)^{1/p}, (\Sigma w_n^r |x_n|^r)^{1/r}\} < \infty$. So the unit vectors of $l\{M_n\}$ are equivalent to the natural basis of $X_{p,r}$. The only Orlicz sequence spaces contained in $l_p \oplus l_r$ are l_p and l_r. So $X_{p,r}$ does not contain any other Orlicz sequence spaces besides l_p and l_r. However, we shall show later in Sec. 4 that $X_{p,r}^*$ contains l_t for every

$t \in [q, s]$, where $(1/p)+(1/q)=1$ and $(1/r)+(1/s)=1$. So $X_{p,r}$ cannot be complemented in $l_p \oplus l_r$, otherwise $l_q \oplus l_s$ would contain $X^*_{p,r}$ and hence every l_t, $t \in [q, s]$.

Now consider $X_{\infty,r}$. Define

$$M_n(x) = \max\{x^n, w_n^r x^r\}$$
$$= \begin{cases} w_n^r x^r & x \in [0, w_n^{r/(n-r)}] \\ x^n & x \geqq w_n^{r/(n-r)}. \end{cases}$$

Then, $\sum M_n((|y_n|)/t) < \infty$ if and only if

$$\max\left\{\left(\sum w_n^r |y_n|^r t^{-r}\right)^{1/r}, \ \sum |y_n|^n t^{-n}\right\} < \infty \text{ for all } t > 0.$$

This implies the unit vector basis of $c\{M_n\}$ is equivalent to the block basis $f_n + w_n g_n$ of $(c\{N_n\} \oplus l_r)_\infty$, where $N_n(x) = x^n$ and $\{f_n\}$, $\{g_n\}$ are the unit vector bases of $c\{N_n\}$ and l_r respectively. It is not hard to show that the unit vector bases of $c\{N_n\}$ and c_0 are equivalent. So the unit vector basis of $c\{M_n\}$ is equivalent to the natural basis of $X_{\infty,r}$.

Reasoning as in the case $p < \infty$, we can show that the only spaces with a symmetric basis contained in $X_{\infty,r}$ are c_0 and l_r, and that $X_{\infty,r}$ is not complemented in $c_0 \oplus l_r$.

4. The spaces $Y_{q,s}$

Let $\infty \geqq p > r > 1$, $(1/p)+(1/q)=1$ and $(1/r)+(1/s)=1$. We are going to construct the dual $Y_{q,s}$ of $X_{p,r}$ in this section, and show that $Y_{q,s}$ is a universal element in $K(s,q)$, i.e. every $l\{M_n\}$ such that $\{M_n\} \in K(s,q)$ is isomorphic to a subspace of $Y_{q,s}$.

First, we compute M_n^*. Suppose $w = \{w_n\}$ satisfies (2) and $w_n \leqq 1$ for all n. Let $p < \infty$. Then

$$M_n^{*\prime}(x) = \begin{cases} \dfrac{x^{s-1}}{r^{s-1} w_n^s} & x \in [0, r w_n^{s/(s-q)}] \\ w_n^{(sq-s)/(s-q)} & x \in [r w_n^{s/(s-q)}, p w_n^{s/(s-q)}] \\ \dfrac{x^{q-1}}{q p^{q-1}} & x \geqq p w_n^{s/(s-q)}. \end{cases}$$

Hence,

$$M_n^*(x)=\begin{cases}\dfrac{x^s}{sr^{s-1}w_n^s} & x\in[0,rw_n^{s/(s-q)}]\\ w_n^{(sq-s)/(s-q)}x-w_n^{sq/(s-q)} & x\in[rw_n^{s/(s-q)},pw_n^{s/(s-q)}]\\ \dfrac{x^q}{qp^{q-1}} & x\geqq pw_n^{s/(s-q)}.\end{cases}$$

M_n^*, being made up of three pieces, is very inconvenient to work with. So we replace $\{M_n^*\}$ by an equivalent sequence of φ-functions $\{N_n\}$.

Let

$$N_n(x)=\min\{x^q,\ w_n^{-s}x^s\}$$

$$=\begin{cases}w_n^{-s}x^s & x\in[0,\ w_n^{s/(s-q)}]\\ x^q & x\geqq w_n^{s/(s-q)}\end{cases}.$$

Then $N_n(x)=sr^{s-1}M_n^*(x)$ on $[0,w_n^{s/(s-q)}]$. $M_n^*(x)/N_n(x)$ is increasing on $[w_n^{s/(s-q)},\ pw_n^{s/(s-q)}]$, so

$$\frac{1}{sr^{s-1}}\leqq\frac{M_n^*(x)}{N_n(x)}\leqq\frac{1}{qp^{q-1}}$$

on $[w_n^{s/(s-r)},\ pw_n^{s/(s-r)}]$. Finally, $N_n(x)=qp^{q-1}M_n^*(x)$ on $[pw_n^{s/(s-r)},\infty)$. Thus

$$qp^{q-1}M_n^*(x)\leqq N_n(x)\leqq sq^{s-1}M_n^*(x)$$

for all $x\geqq 0$. By Proposition 2.3, $\{M_n^*\}\sim\{N_n\}$.

Now consider the case $p=\infty$.

$$M_n^*(x)=\begin{cases}\dfrac{x^s}{sr^{s-1}w_n^s} & x\in[0,\ rw_n^{(sn-s)/(sn-s-n)}]\\ w_n^{s/(sn-s-n)}x-w_n^{sn/(sn-s-n)} & x\in[rw_n^{(sn-s)/(sn-s-n)},\ nw_n^{(sn-s)/(sn-s-n)}]\\ (n-1)n^{-n/(n-1)}x^{n/(n-1)} & x\geqq nw_n^{(sn-s)/(sn-s-n)}.\end{cases}$$

In this case, it is imperative to replace M_n^* by something more civilized. Let

$$N_n(x)=\min\{x,\ w_n^{-s}x^s\}$$

$$=\begin{cases}w_n^{-s}x^s & x\in[0,\ w_n^{s/(s-1)}]\\ x & x\geqq w_n^{s/(s-1)}.\end{cases}$$

Then $N_n(x)=sr^{s-1}M_n^*(x)$ on $[0,w_n^{s/(s-1)}]$. $M_n^*(x)/N_n(x)$ is increasing on $[w_n^{s/(s-1)},\ 1]$. So

$$\frac{1}{sr^{s-1}} \leqq \frac{M_n^*(x)}{N_n(x)} \leqq (n-1)n^{-n/(n-1)} \leqq \frac{n-1}{n} \leqq 1$$

on $[w_n^{s/(s-1)}, 1]$. Thus,

$$M_n^*(x) \leqq N_n(x) \leqq sr^{s-1} M_n^*(x)$$

for all $x \in [0, 1]$. As $\inf M_n^*(1) > 0$, $\inf N_n(1) > 0$, Proposition 2.3 (a) implies $\{M_n^*\} \sim \{N_n\}$.

Note that for $1 < q < 2$, $Y_{2,q}$ is the space X_q of [10].

We now generalize Rosenthal's result that X_q contains l_t for all $t \in [q, 2]$. We want to show that $Y_{q,s}$ contains l_t for all $t \in [q, s]$. As we no longer have $s = 2$, we cannot use anything involving L_p-spaces. So we have to have new techniques. One thing comes to mind immediately. Lindenstrauss and Tzafriri have proved in Theorem 1 of [5] that every Orlicz sequence space l_M contains l_t for every t in the interval of M. We want to apply the techniques of that proof.

The proof in [5] is as follows. Suppose t is in the interval of M. Let

$$G_n(x) = \frac{1}{C_n} \int_{a_n}^{1} \frac{M(b_n x u)}{u^{t+1}}\, du,$$

where $\{a_n\}$, $\{b_n\}$ are sequences of positive numbers satisfying certain convergence properties, and

$$C_n = \int_{a_n}^{1} \frac{M(b_n u)}{u^{t+1}}\, du.$$

Then $\{G_n(x)\}$ is shown to converge pointwise to x^t. As

$$G_n \in C_{M,b_n} = \overline{\operatorname{conv}} \left\{ \frac{M(sx)}{M(s)} : 0 < s \leqq b_n \right\}, \quad x^t \in \bigcap_{s>0} C_{M,s}.$$

By Th. 1 of [4], l_M contains a subspace isomorphic to l_t.

We imitate this argument for $l\{N_n\}$. We put

$$G_n(x) = \frac{1}{C_n} \int_{a_n}^{1} \frac{N_n(b_n x u)}{u^{t+1}}\, du,$$

and then prove that $\{G_n(x)\}$ converges pointwise to x^t. However, we do not have any result corresponding to Th. 1 of [4]. In fact, no such general result seems to be possible for modular sequence spaces. However, for the special case of $l\{N_n\}$, we do have something similar. This is the motivation for Proposition 4.2.

First note that we no longer have to distinguish between the cases $p < \infty$ and $p = \infty$. For $pr/(p-r) = sq/(s-q)$, while $r = s/(s-1)$. So (2) can be written as $\lim w_n = 0$, $\Sigma w_n^{sq/(s-q)} = \infty$. We also require $w_n \leqq 1$. So let (4) be

$$w_n \in (0,1], \quad \lim w_n = 0, \quad \sum w_n^{sq/(s-q)} = \infty. \tag{4}$$

LEMMA 4.1. *Suppose* $\alpha \in (0,1]$, $\beta > 0$, *and* $u \in (0,1]$. *Let* $N_u(x) = \min\{u^{-s}x^s, x^q\}$. *Then, there exist* $\gamma > 0$ *and* $v \in (0,1]$ *such that*

$$\alpha N_u(\beta x) = N_v(\gamma x),$$

where $N_v(x) = \min\{v^{-s}x^s, x^q\}$.

PROOF. $\alpha N_u(\beta x) = \min\{\alpha u^{-s}\beta^s x^s, \alpha\beta^q x^q\}$. Put $\gamma = \alpha^{1/q}\beta$ and $v = \alpha^{(1/q)-(1/s)}u$. Since $(1/q)-(1/s) > 0$, and $\alpha \in (0,1]$, $v \in (0,1]$. Then

$$\begin{aligned} N_v(\gamma x) &= \min\{v^{-s}\gamma^s x^s, \gamma^q x^q\} \\ &= \min\{\alpha u^{-s}\beta^s x^s, \alpha\beta^q x^q\} = \alpha N_u(\beta x). \end{aligned}$$

Q.E.D.

PROPOSITION 4.2. *Let* $w = \{w_n\}$ *be a sequence satisfying* $w_n \in (0,1]$ *and* $\lim w_n = 0$. *Let* $N_{w,n}$ *be the* φ*-function* $\min\{w_n^{-s}x^s, x^q\}$. *Suppose* $\{f_n\}$ *is a sequence of nonnegative continuous functions. Put*

$$G_n(x) = \frac{1}{C_n}\int_{a_n}^{1} N_{w,n}(b_n x u) f_n(u)\,du,$$

where $a_n, b_n > 0$, *and*

$$C_n = \int_{a_n}^{1} N_{w,n}(b_n u) f_n(u)\,du.$$

Then G_n *is a* φ*-function between* s *and* q. *Moreover, we have a sequence of positive numbers* $v = \{v_n\}$, *with* $\lim v_n = 0$ *and* $v_n \leqq 1$, *and* φ*-functions*

$$H_n(x) = \sum_{i=k_n+1}^{k_{n+1}} N_{v,i}(\gamma_i x),$$

where $k_1 < k_2 < \cdots$ *are positive integers, such that*

$$\sum_{n=1}^{\infty} \sup_{x \in [0,1]} |G_n(x) - H_n(x)| < \infty.$$

Hence $Y_{q,s}$ *contains a subspace isomorphic to* $l\{G_n\}$.

PROOF. It is trivial to show that G_n is a φ-function between s and q.

The Riemann sums of

$$\int_{a_n}^{1} N_{w,n}(b_n xu) f_n(u) du$$

converge uniformly with respect to x on [0, 1] by Dini's Theorem. So given n, we have a partition $a_n = u_0 < u_1 < \cdots < u_{j_n} = 1$ such that for all $x \in [0, 1]$,

$$\left| G_n(x) - \sum_{i=1}^{j_n} C_n^{-1} N_{w,n}(b_n x u_i) f_n(u_i)(u_i - u_{i-1}) \right| < 2^{-n}. \tag{5}$$

Put $\beta_{n,i} = b_n u_i$, and $\alpha_{n,i} = C_n^{-1} f_n(u_i)(u_i - u_{i-1})$. Since f_n is bounded on $[a_n, 1]$, and C_n is a constant, by taking the partition fine enough, we can assume $\alpha_{n,i} \leqq 1$. So we can apply Lemma 4.1 to obtain $\gamma_{n,i}$ and $v_{n,i} \in (0,1]$ such that

$$C_n^{-1} f_n(u_i)(u_i - u_{i-1}) N_{w,n}(b_n x u_i) = N^{*}_{n,i}(\gamma_{n,i} x), \tag{6}$$

where $N^{*}_{n,i}(x) = \min\{v_{n,i}^{-s} x^s, x^q\}$. Let $k_1 = 0$, and $k_n = \Sigma_{i=1}^{n-1} j_i$ for $n > 1$. If $k_n < l \leqq k_{n+1}$, let $v_l = v_{n,l-k_n}$, $\gamma_l = \gamma_{n,l-k_n}$, and $N_{v,l} = N^{*}_{n,l-k_n}$. Thus by the definition of $v_{n,i}$ in Lemma 4.1, $\lim v_l = 0$. (5) and (6) imply that

$$\left| G_n(x) - \sum_{l=k_n+1}^{k_{n+1}} N_{v,l}(\gamma_l x) \right| < 2^{-n}. \tag{7}$$

Putting $H_n(x) = \Sigma_{l=k_n+1}^{k_{n+1}} N_{v,l}(\gamma_l x)$, we have the desired result.

By Proposition 2.3 (b), $\{G_n\} \sim \{H_n\}$. As $l\{H_n\}$ is isomorphic to a subspace of $l\{N_{v,l}\}$, so is $l\{G_n\}$. Finally, $l\{N_{v,l}\}$ is either l_q or $Y_{s,q}$. For if $\Sigma v_l^{sq/(s-q)} < \infty$, then $X_{p,r,v} \sim l_p$, and so $l\{N_{v,l}\} \sim l_q$, which is contained in $Y_{q,s}$. If $\Sigma v_l^{sq/(s-q)} = \infty$, then $v = \{v_l\}$ satisfies (2), and $l\{N_{v,l}\} \sim Y_{s,q}$ by Theorem 3.3. Q.E.D.

COROLLARY 4.3. *Let $w = \{w_{m,n}\}$ be a double sequence of positive numbers satisfying $w_{m,n} \leqq 1$, and $\lim_{m,n\to\infty} w_{m,n} = 0$. Let $f_{m,n}, G_{m,n}$ be as in Proposition 4.2. Suppose for each m and $x \in [0, 1]$,*

$$\lim_{m\to\infty} G_{m,n}(x) = G_m(x).$$

Then $Y_{q,s}$ contains a subspace isomorphic to $l\{G_m\}$.

PROOF. $\{G_{m,n}\} \in K(s,q)$ by Proposition 4.2, and $\{G_{m,n}\}$ is normalized. By Theorem 2.5 (b), for each m, there is an $n(m)$ such that

$$\sup_{x\in[0,1]} |G_{m,n(m)}(x) - G_m(x)| < 2^{-m}.$$

By Proposition 4.2, we have $H_{m,n(m)}$ of the form $\Sigma_{n\in E_m} N_{v,n}(\gamma_n x)$, where $\{E_m\}$ are disjoint finite subsets of Z^+, such that

$$\sum_{m=1}^{\infty} \sup_{x \in [0,1]} |G_{m,n(m)}(x) - H_{m,n(m)}(x)| < \infty.$$

So $\sum_{m=1}^{\infty} \sup_{x \in [0,1]} |G_m(x) - H_{m,n(m)}(x)| < \infty$ and $\{G_m\} \sim \{H_{m,n(m)}\}$. This immediately shows that $l\{G_m\}$ is isomorphic to a subspace of $Y_{q,s}$.

REMARK. We can say more about the isomorphic imbedding of $l\{G_m\}$ into $Y_{q,s}$, where $l\{G_m\}$ is as in the corollary. Fix $w = \{w_n\}$ satisfying (4). Suppose $v = \{v_n\}$ is another sequence satisfying (4). Then by Theorem 3.3, the natural basis of $X_{p,r,v}$ is equivalent to a block basis of the natural basis of $X_{p,r,w}$, and the span of the block basis is complemented in $X_{p,r,w}$. This implies that the dual basis of $X_{p,r,v}$ is also a block basis of the dual basis of $X_{p,r,w}$. In other words, the unit vector basis of $l\{N_{v,n}\}$ is equivalent to a block basis of the unit vector basis of $l\{N_{w,n}\}$. So the unit vector basis of $l\{G_m\}$ is equivalent to a block basis of the unit vector basis of $l\{N_{w,n}\} = Y_{q,s}$.

We can now prove the result we promised.

PROPOSITION 4.4. *l_t is isomorphic to a subspace of $Y_{q,s}$ if and only if $q \leqq t \leqq s$.*

PROOF. "only if":

$\{N_n\} \in K(s,q)$. So by Proposition 2.6 (b), $t \in [q,s]$.

"if":

We already proved that l_p and l_r are complemented subspaces of $X_{p,r}$. So l_q and l_s are complemented subspaces of $Y_{q,s}$. Let $t \in (q,s)$.

Take $a_n \in (0,1]$, $a_n \to 0$, $b_n = a_n^{(t-q)/t}$, and $w_n = a_n^{(s-q)/s}$. Then $w_n \to 0$ and $w_n \leqq 1$. Let

$$G_n(x) = \frac{1}{C_n} \int_{a_n}^{1} \frac{N_n(b_n x u)}{u^{t+1}}\, du,$$

where

$$C_n = \int_{a_n}^{1} \frac{N_n(b_n u)}{u^{t+1}}\, du.$$

Our aim of course is to prove $\lim_{n\to\infty} G_n(x) = x^t$ for all x, and so by Corollary 4.3, l_t can be imbedded in $Y_{q,s}$.

$$G_n(x)=\frac{1}{C_n}\int_{a_n}^{a_n/b_nx} a_n^{q-s}b_n^s x^s u^{s-t-1}\,du$$

$$+\frac{1}{C_n}\int_{a_n/b_nx}^{1} b_n^q x^q u^{q-t-1}\,du$$

$$=\frac{1}{C_n}\,[(s-t)^{-1}a_n^{q-s}b_n^s x^s u^{s-t}]_{a_n}^{a_n/b_nx}$$

$$+\frac{1}{C_n}\,[(q-t)^{-1}b_n^q x^q u^{q-t}]_{a_n/b_nx}^{1}$$

$$=\frac{1}{C_n}\left[\frac{s-q}{(s-t)(t-q)}x^t+\frac{b_n^q x^q}{q-t}-\frac{x^s}{s-t}\,a_n^{t^{-1}(s-t)(t-q)}\right].$$

Put $x=1$, and let $n\to\infty$. Then

$$\lim_{n\to\infty} C_n=\frac{s-q}{(s-t)(t-q)},$$

since b_n^q and $a_n^{t^{-1}(s-t)(t-q)}$ both converge to 0. So for all x,

$$\lim_{n\to\infty} G_n(x)=x^t. \qquad \text{Q.E.D.}$$

We now prove that every Orlicz sequence space l_F, where $F\in K(s,q)$, can be imbedded into $Y_{q,s}$. The proof is a modification of the above proof, and is inspired by the proof of Theorem 3.1 in [12].

THEOREM 4.5. *Let F be an Orlicz function, and let $1\leqq q<s<\infty$. Then l_F is isomorphic to a subspace of $Y_{q,s}$ if and only if F is equivalent to an Orlicz function $G\in K(s,q)$.*

PROOF. "only if":
This follows immediately from Proposition 2.6 (b).

"if":
We have already proved l_q, l_s are isomorphic to subspaces of $Y_{q,s}$. So assume $F\in K(s,q)$, $F(1)=1$, and F not equivalent to x^q or x^s. By Corollary 2.2, we can further assume that $F(x)/x^q$ is a φ-function. Put $P(x)=x^{-(q/(s-q))}F(x^{1/(s-q)})$. Then P is a φ-function. Also, $P(x)/x=x^{-(s/(s-q))}F(x^{1/(s-q)})$ is a decreasing function. So by Theorem 2.1 (b), there exists a concave φ-function Q, with continuous second derivative, such that

$$4P(x)\geqq Q(x)\geqq P(x)\quad\text{for all}\quad x\geqq 0. \tag{8}$$

Let $a_n\to 0$, $b_n=a_n^{(2s-2q)/(2s-q)}$, and $w_n=a_n^{(s-q)/s}$. Let

$$G_n(x)=\frac{1}{C_n}\int_{a_n}^{1}-(s-q)\,u^{q-1-2s}N_n(b_nxu)\,Q''(a_n^{s-q}b_n^{q-s}u^{q-s})du,$$

where

$$C_n=\int_{a_n}^{1}-(s-q)\,u^{q-1-2s}N_n(b_nu)\,Q''(a_n^{s-q}b_n^{q-s}u^{q-s})du.$$

As Q'' is non-positive and continuous, the hypothesis of Proposition 4.2 is satisfied. So we only have to prove that $\{G_n\}$ converges pointwise to a function equivalent to F.

$$G_n(x)=\frac{1}{C_n}\int_{a_n}^{a_n/xb_n}-(s-q)a_n^{q-s}b_n^{s}u^{q-s-1}x^{s}Q''(a_n^{s-q}b_n^{q-s}u^{q-s})du$$

$$+\frac{1}{C_n}\int_{a_n/xb_n}^{1}-(s-q)b_n^{q}u^{2q-2s-1}x^{q}Q''(a_n^{s-q}b_n^{q-s}u^{q-s})du.$$

Put $v=a_n^{s-q}b_n^{q-s}u^{q-s}$. Then

$$dv=-(s-q)a_n^{s-q}b_n^{q-s}u^{q-s-1}du,$$

and

$$G_n(x)=\frac{x^s}{C_n}\int_{b_n{}^{q-s}}^{x^{s-q}}a_n^{2q-2s}b_n^{2s-q}Q''(v)dv$$

$$+\frac{x^q}{C_n}\int_{x^{s-q}}^{a_n{}^{s-q}b_n{}^{q-s}}a_n^{2q-2s}b_n^{2s-q}v\,Q''(v)dv$$

$$=\frac{x^s}{C_n}\int_{b_n{}^{q-s}}^{x^{s-q}}Q''(v)dv+\frac{x^q}{C_n}\int_{x^{s-q}}^{a_n{}^{s-q}b_n{}^{q-s}}v\,Q''(v)dv$$

$$=\frac{x^s}{C_n}[Q'(x^{s-q})-Q'(b_n^{q-s})]$$

$$+\frac{x^q}{C_n}[a_n^{s-q}b_n^{q-s}Q'(a_n^{s-q}b_n^{q-s})-x^{s-q}Q'(x^{s-q})]$$

$$+\frac{x^q}{C_n}\int_{a_n{}^{s-q}b_n{}^{q-s}}^{x^{s-q}}Q'(v)dv$$

$$=-C_n^{-1}x^sQ'(b_n^{q-s})+C_n^{-1}x^qa_n^{s-q}b_n^{q-s}Q'(a_n^{s-q}b_n^{q-s})$$

$$+C_n^{-1}x^qQ(x^{s-q})-C_n^{-1}x^qQ(a_n^{s-q}b_n^{q-s}).$$

Put $x=1$, and let $n\to\infty$. Q is concave. So $Q(x)/x\geqq Q'(x)\geqq 0$. Q is a φ-function, so $\lim_{x\to 0}Q(x)=0$ and $\lim_{x\to 0}xQ'(x)=0$. Also, we can assume $\lim_{x\to\infty}Q'(x)=0$, as the behaviour at ∞ is irrelevant. Noting that

$$a_n^{s-q} b_n^{q-s} = a_n^{(q(s-q))/(2s-q)} \to 0,$$

we have $\lim_{n\to\infty} C_n = Q(1)$. So

$$\lim_{n\to\infty} G_n(x) = \frac{x^q}{Q(1)} Q(x^{s-q}).$$

Denote this limit by $G(x)$. By Corollary 4.3, $Y_{q,s}$ contains a subspace isomorphic to l_G.

By (8),

$$\frac{1}{Q(1)} x^q P(x^{s-q}) \leqq G(x) \leqq \frac{4x^q}{Q(1)} P(x^{s-q}) \text{ for all } x \geqq 0.$$

Substituting $P(x^{s-q}) = F(x)/x^q$, we have

$$\frac{F(x)}{Q(1)} \leqq G(x) \leqq \frac{4F(x)}{Q(1)} \text{ for all } x \geqq 0.$$

Since $F(1) = G(1) = 1$, $Q(1)$ must be between 1 and 4.

Hence

$$\tfrac{1}{4} F(x) \leqq G(x) \leqq 4F(x) \text{ for all } x \geqq 0. \tag{9}$$

So $F \sim G$ and $Y_{q,s}$ contains a subspace isomorphic to l_F. Q.E.D.

REMARK. Combining Proposition 4.2 and Theorem 4.5, we have the following representation of an Orlicz function F. Every Orlicz function $F \in K(s,q)$ for some $s < \infty$ is equivalent to a uniform limit of functions of the form $\Sigma_{i=p_n+1}^{p_{n+1}} N_i(a_i x)$, where $N_i(x) = \min\{w_i^{-s} x^s, x^q\}$, $w = \{w_i\}$ satisfies (4), $a_i > 0$, $p_1 < p_2 < \cdots$, and $\Sigma_{i=p_n+1}^{p_{n+1}} N_i(a_i) = 1$.

COROLLARY 4.6. *Let $\{F_m\}$ be a sequence of Orlicz functions and $\{F_m\} \in K(s,q)$ for some $s < \infty$. Then $Y_{q,s}$ contains a subspace isomorphic to $l\{F_m\}$.*

PROOF. Without loss of generality, assume $\{F_m\}$ to be normalized and assume F_m is not equivalent to x^q for all m. By Corollary 2.2, we can assume $x^{-q}F_m(x)$ to be a φ-function for all m. Then by Theorem 4.5, we have

$$G_{m,n}(x) = \frac{1}{C_{m,n}} \int_{a_{m,n}}^{1} N_{m,n}(b_{m,n} xu) f_{m,n}(u) du$$

such that $\lim_{n\to\infty} G_{m,n}(x) = G_m(x)$ for all $x \geqq 0$, and

$$\tfrac{1}{4} F_m(x) \leqq G_m(x) \leqq 4F_m(x) \text{ for all } x \geqq 0.$$

So $\{G_m\} \sim \{F_m\}$. As $l\{G_m\}$ can be imbedded into $Y_{q,s}$ by Corollary 4.3, so can $l\{F_m\}$.

REMARKS. (i) Corollary 4.6 can also be proved as follows: Lindenstrauss showed that by an argument similar to that of Proposition 3 in [4], every modular sequence space $l\{F_m\}, \{F_m\} \in K(s,q)$, can be imbedded into an Orlicz sequence space l_F, where $F \in K(s,q)$. So by Theorem 4.5, $l\{F_m\}$ is isomorphic to a subspace of $Y_{q,s}$. (ii) Unlike Theorem 4.5, the necessity part of Corollary 4.6 is false. This is a consequence of the following example:

EXAMPLE. Suppose $F_i(x) = x^{p_i}$ for all i. Then $l\{F_i\}$ is called a Nakano space, which was studied by Nakano in [9]. Let $s \in [1, 2)$, $b_n > 0$, and

$$\sum_{n=1}^{\infty} b_n (1 + b_n)^{-1-(1/(b_n))} = \epsilon.$$

Suppose $k_1 < k_2 < \cdots$ are positive integers, and $p_i = s(1 + b_n)$ for all $k_n < i \leqq k_{n+1}$. We are going to show that the Nakano space $l\{F_i\}$ can be imbedded in l_s. Moreover, if

$$(k_{n+1} - k_n) \geqq n^{\epsilon b_n^{-1} - 1}$$

then the unit vector basis of $l\{F_i\}$ is not s-dominating. This shows that for $s \in [1, 2)$, $l\{F_i\}$ can be imbedded into $Y_{q,s}$, but $\{F_i\}$ is not equivalent to any sequence in $K(s,q)$.

Let l_p^n be the n-dimensional space with the p-norm. By our choice of b_n and k_n, it is not hard to show that $l\{F_i\}$ and $(\Sigma \oplus l_{s(1+b_n)}^{k_{n+1}-k_n})_s$ are $(1+\epsilon)^{1/s}$-isomorphic. As $s < 2$ and $b_n \to 0$, we can assume $s(1 + b_n) \leqq 2$ for all n. So $l_{s(1+b_n)}$ can be isometrically imbedded into L_s for $s \in [1, 2)$. L_s is an $\mathscr{L}_{s,1+\delta}$ space for all $\delta > 0$. This immediately shows that $(\Sigma \oplus l_{s(1+b_n)}^{k_{n+1}-k_n})_s$ can be imbedded into l_s.

Finally, suppose $(k_{n+1} - k_n) \geqq n^{\epsilon b_n^{-1} - 1}$. We are going to show that there is some $\{a_i\} \in l\{F_i\}$ such that $\Sigma |a_i|^s = \infty$. In fact, we can simply take $a_i = n^{-1/s}(k_{n+1} - k_n)^{-1/s}$ for $k_n < i \leqq k_{n+1}$. On the other hand, if $\{G_i\} \in K(q,s)$ and $\{x_i\} \in l\{G_i\}$, then $\Sigma |x_i|^s < \infty$. So $\{F_i\}$ cannot be equivalent to any $\{G_i\} \in K(s,q)$.

REMARKS. (i) $Y_{q,s}$ is a quotient of $l_q \oplus l_s$. Thus the class of subspaces of quotients of $l_q \oplus l_s$ contains all $l\{F_m\}$, where $\{F_m\} \in K(s,q)$. In particular, it contains every l_t, $t \in [q,s]$. So it is not possible to generalize Johnson and Zippin's result in [2] to the spaces $l_q \oplus l_s$.

(ii) Theorem 4.5 shows that every separable Orlicz sequence space is a quotient space of $X_{p,r}$, for some $1 < r < p \leqq \infty$.

(iii) The space $Y_{q,2}$ is just the space X_q of [10], for $q > 1$. Proposition 4.4 then becomes the same as Cor. 4.2 of [11], although our proof does not use probability theory. Theorem 4.5, together with corollary to Th. 4 of [10], gives an imbedding of every Orlicz sequence space l_F, $F \in K(2,q)$, $2 > q > 1$, into L_q, and we only use probability theory for the imbedding of X_q into L_q.

In [11], the imbedding of l_F into X_q is obtained by another way. Th. IV.3 of [1] shows that every Orlicz sequence space l_F, $F \in K(2,q)$, $2 > q > 1$, can be imbedded into L_q as the span of a sequence of independent random variables. By Cor. 4.1 of [11], l_F can be imbedded into X_q.

(iv) It would be desirable to obtain a direct imbedding of $Y_{1,2}$ into L_1. For by the remark at the end of Corollary 4.6, $Y_{1,2}$ can be imbedded into an Orlicz sequence space l_F, where $F \in K(1,2)$. By Th. IV.3 of [1], l_F can be imbedded into L_1. However, this method seems rather involved.

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